Rings and Modules Final Test
(BMath-2nd year, 2024)

Instructions: Total time 3 Hours. Solve problems (even partially), for a maximum score of 50. Use terminologies, notations and results as covered in the course, no need to prove such results. If you wish to use a problem given in an assignment or homework or any other such, please supply its full solution.

1. Let R be a ring and M be a Noetherian R-module and $f \in \mathrm{End}_R(M)$ be surjective. Prove that f is injective. (10)

2. Let R be a ring and M, N, S be R-modules such that $M \oplus S \cong N \oplus S$. Can we conclude from this that $M \cong N$ as R-modules? Explain. (10)

3. Let k be a field and $V = k^2$, $T \in \mathrm{End}_k(V)$. Prove that every $v \in V$ which is not an eigenvector for T, is a cyclic vector for (V, T). (10)

4. Let
$$A = \begin{pmatrix} 2 & -2 & 6 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{pmatrix} \in M_3(\mathbb{R})$$
and $V = \mathbb{R}^3$. Write down the invariant factors decomposition of V as an $\mathbb{R}[x]$ module via the module structure given by $x.v := Av$ for v a column vector in V. Compute the Jordan canonical form of A. (7+3)

5. Prove that any non-cyclic finite abelian group G contains a subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ for some prime p. (5)

6. Let
$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{C}).$$
Compute the invariant factors decomposition and the elementary divisors decomposition of $(\mathbb{C}^2, A)$. Compute the Jordan form for A. (3+2)

7. Let $\mathcal{Z} = \{A \in \mathrm{GL}_2(\mathbb{F}_q) | \chi_A(x) = m_A(x) \text{ is reducible}\}$, where q is a power of an odd prime, χ_A, m_A are respectively the characteristic and the minimal polynomials of A. Compute the number of elements in $\mathcal{Z}$. (10)

8. Let V be a finite dimensional vector space over a field k. Let $T \in \mathrm{End}_k(V)$ be such that $\mathrm{rank}(T) = 1$. Prove that either T is nilpotent or diagonalizable, but not both. (5)